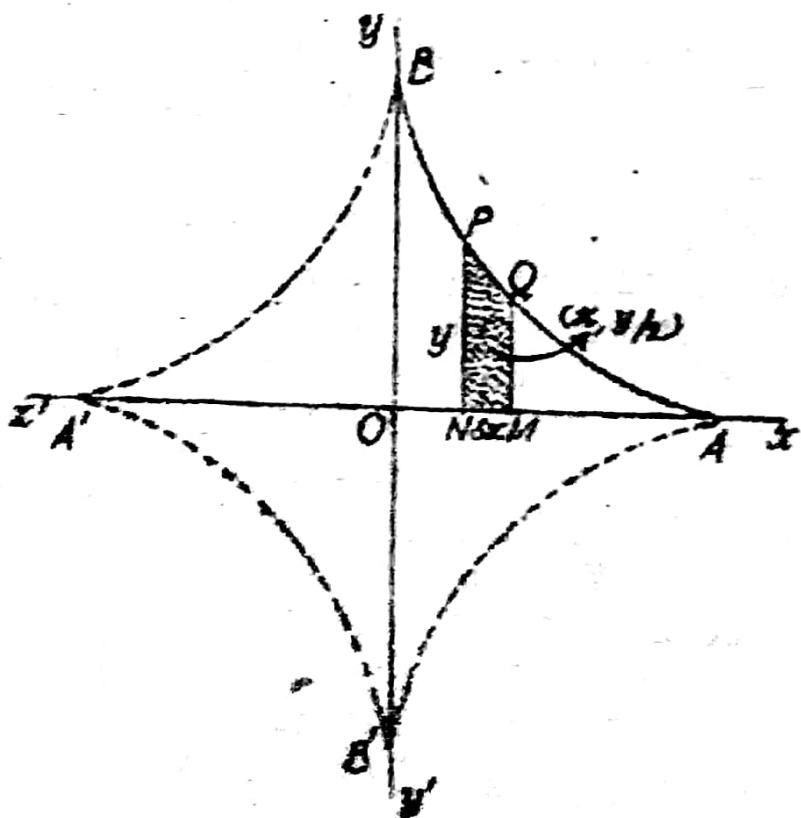


$$\text{H-10} \\ \cancel{\text{R-105}} = 2a \frac{(5/6) \cdot \frac{3}{4} \cdot \frac{1}{2} \pi}{\frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} \pi} = \frac{5a}{3}$$

Hence $\bar{x} = \frac{5a}{3}$, $\bar{y} = 0$

Ex. 11. Find the position of C.G. of the area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the positive quadrant.

(Agra 86; Gorakhpur 89; Rohilkhand 86)



Solution. $OABO$ is the positive quadrant. Take two points $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ on the arc AB . From P and Q draw PN and QM perpendiculars to x -axis.

Area $PNMQ$ is the elementary area. Its area = $y \delta x$
and mass = $\delta m = \rho \cdot y \delta x$,

where ρ is the mass per unit area. Also C.G. of this area is $(x, \frac{1}{2}y)$.

Also the curve is symmetrical about the line $y=x$... (i)

Hence the required C.G. will also lie on $y=x$, i.e. if (\bar{x}, \bar{y}) be the required C.G., then $\bar{x}=\bar{y}$.

$$\text{Also } \bar{x} = \frac{\int x dm}{\int dm} = \frac{\int_{x=0}^a x \rho y dx}{\int_{x=0}^a \rho y dx}, \text{ where } y^{2/3} = a^{2/3} - x^{2/3}$$

Formula

$$\text{If } J_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx, \text{ then}$$

$$J_{m,n} = \frac{n-1}{m+n} J_{m,n-2} = \frac{m-1}{m+n} J_{m-2,n}$$

$$\therefore J_{5,4} = \int_0^{\pi/2} \sin^5 x \cos^4 x dx$$

$$= \frac{3}{9} \cdot J_{5,2}$$

$$= \frac{3}{9} \cdot \frac{2-1}{5+2} J_{5,0}$$

$$= \frac{3}{9} \cdot \frac{1}{7} \cdot \int_0^{\pi/2} \sin^5 x \cos^0 x dx$$

$$= \frac{3}{9} \cdot \frac{1}{7} \cdot \int_0^{\pi/2} \sin^5 x dx$$

$$= \frac{3}{9} \cdot \frac{1}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

$$= \frac{8}{9 \cdot 7 \cdot 5}$$

$$J_{2,4} = \frac{3}{6} \cdot J_{2,2}$$

$$= \frac{3}{6} \cdot \frac{2-1}{4} \cdot J_{2,0}$$

$$= \frac{3}{6} \cdot \frac{1}{4} \cdot \int_0^{\pi/2} \sin^2 x \cos^0 x dx$$

$$= \frac{3}{6} \cdot \frac{1}{4} \cdot \int_0^{\pi/2} \sin^2 x dx$$

$$= \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{\pi}{32}$$

$$\therefore \bar{x}_2 = a \cdot \frac{8/9 \cdot 7 \cdot 5}{7! \cdot 3!} = \cancel{\frac{8}{7!}} = \cancel{\frac{8}{7!}} \cdot \frac{256 \pi}{315 \pi} \quad \checkmark$$

M-II'97 Hence the required C.G. is $\left[\frac{a(9\pi^2 - 16)}{18\pi}, \frac{7a}{6} \right]$. Ans.

M-16 H.D'92 **Ex. 16. Find the position of the centroid of the area of the cardioid $r = a(1 + \cos \theta)$. (Agra 87; Purvanchal 89)

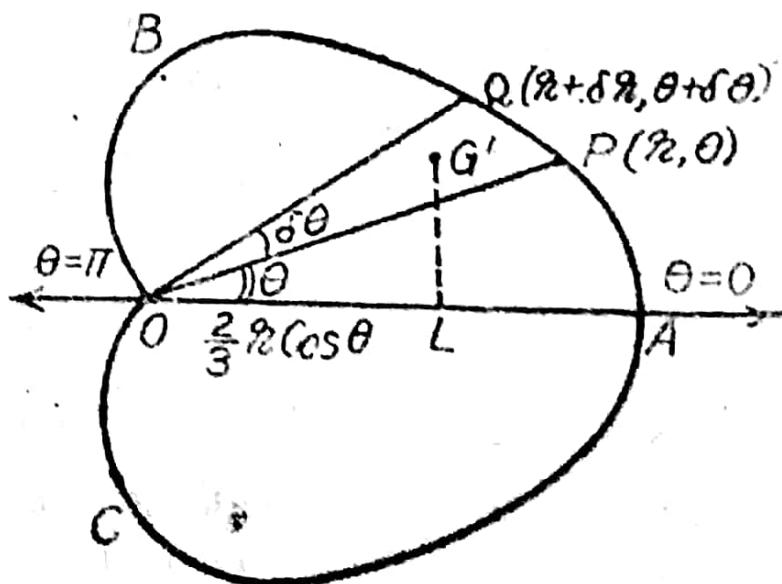
Solution. Since the curve is symmetrical about the initial line, so $\bar{y} = 0$. Also for the whole cardioid θ varies from $-\pi$ to π .

Take two neighbouring points $P(r, \theta)$ and

$Q(r + \delta r, \theta + \delta \theta)$

on the curve. The elementary area is the sector POQ . Its area $= \frac{1}{2}r^2\delta\theta$ and x -coordinate of its C.G. is $OL = (2/3)r \cos \theta$.

$\therefore \bar{x}$, the x -coordinate of the C.G. of the car-



(Fig. 20)

diod is given by

$$\bar{x} = \frac{\int x dm}{\int dm} = \frac{\int_0^\pi (2/3) r \cos \theta \frac{1}{2} r^2 d\theta}{\int_0^\pi \frac{1}{2} r^2 d\theta} = \frac{\int_0^\pi (2/3) r^2 \cos \theta d\theta}{\int_0^\pi r^2 d\theta}$$

$$= \frac{\frac{2}{3} \int_0^\pi a^2 (1+\cos \theta)^2 \cos \theta d\theta}{\int_0^\pi a^2 (1+\cos \theta)^2 d\theta},$$

$$\therefore \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(x) = f(-x)$$

$$= \frac{2a \int_0^\pi (1+\cos \theta)^2 \cos \theta d\theta}{3 \int_0^\pi (1+\cos \theta)^2 d\theta} \quad \dots (i)$$

Numerator of $\bar{x} = 2a \int_0^\pi (1+\cos \theta)^2 \cos \theta d\theta$

$$= 2a \int_0^\pi (2 \cos \frac{1}{2}\theta)^2 (2 \cos^2 \frac{1}{2}\theta - 1) d\theta$$

$$= 2a \int_0^{\pi/2} (2 \cos^2 \phi)^2 (2 \cos^2 \phi - 1) 2 d\phi, \text{ putting } \theta = 2\phi$$

$$= 64a \int_0^{\pi/2} \cos^8 \phi d\phi - 32a \int_0^{\pi/2} \cos^6 \phi d\phi$$

$$= 64a \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - 32a \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{15\pi a}{4}$$

And denominator of $\bar{x} = 3 \int_0^\pi (1+\cos \theta)^2 d\theta = 12 \int_0^{\pi/2} \cos^4 \frac{1}{2}\theta d\theta$

$$= 12 \int_0^{\pi/2} \cos^4 \phi \cdot 2 d\phi, \text{ putting } \theta = 2\phi$$

$$= 24 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{9\pi}{2}$$

$$\therefore \text{From (i), } \bar{x} = \frac{15\pi a/4}{9\pi/2} = \frac{5a}{6}.$$

Hence the required C.G. is given by $\bar{x} = (5/6) a, \bar{y} = 0$. Ans.

Ex. 17. Find the coordinates of the centre of gravity of the area of the loop of the curve $r = a \sin 2\theta$ which lies in the positive quadrant, the density being supposed uniform. (Agra 85)

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Ex. 5. Find the centroid of the arc of the curve $x=a \cos^3 t$, $y=a \sin^3 t$ lying in the first quadrant.

Solution. Refer Fig. 17 Page 14.

We are to find the C. G. of the arc AB , which extends from A where $t=0$ to B where $t=\pi/2$.

Also the curve is symmetrical about the line $y=x$. i.e. $t=\sqrt{\frac{\pi}{4}}$

The required C. G. will also lie on $y=x$ i.e. If (\bar{x}, \bar{y}) be the required C. G., then $\bar{x}=\bar{y}$.

$$\text{Now we know that } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2}.$$

$$\therefore x=a \cos^3 t, y=a \sin^3 t \text{ (given)}$$

$$= 3a \sin t \cos t, \text{ on simplifying}$$

$$\text{or } ds = 3a \sin t \cos t dt \quad \dots \text{(ii)}$$

Now consider an elementary arc $PQ=\delta s$. Its mass $= \rho \delta s$, where ρ is the mass per unit length of the arc and its C.G. is (x, y)

If (\bar{x}, \bar{y}) be the required C.G. of the arc AB , then

$$\bar{x} = \frac{\int x dm}{\int dm} = \frac{\int_{t=0}^{\pi/2} x \rho ds}{\int_{t=0}^{\pi/2} \rho ds} = \frac{\int_0^{\pi/2} a \cos^3 t \cdot 3a \sin t \cos t dt}{\int_0^{\pi/2} 3a \sin t \cos t dt}$$

$$= \frac{a \int_0^{\pi/2} \sin t \cos^4 t dt}{\int_0^{\pi/2} \sin t \cos t dt} = \frac{-a \int_0^0 z^4 dz}{-\int_1^0 z dz}, \text{ where } \cos t = z$$

$$= \frac{a (z^5/5)_0^1}{(z^2/2)_0^1} = \frac{2a}{5}.$$

\therefore From (i), the required C.G. is $\bar{x}=(2/5) a = \bar{y}$.

Ans.

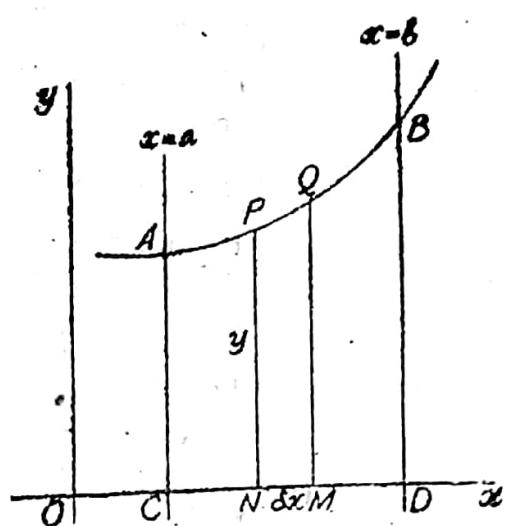
§ 8. C.G. of a Solid of Revolution.

Let the solid be generated by the revolution of the curve $y=f(x)$ lying between ordinates $x=a$ and $x=b$ about x-axis.

Take two neighbouring points $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ on the curve. From P and Q draw PN and QM perpendiculars to x-axis. If this area $PNMQ$ is revolved about x-axis, it would generate a disc whose volume $=\pi y^2 \delta x$ and the C.G. of the disc may be supposed to be $N(x, 0)$, since $NM=\delta x$ is very small.

\therefore If ρ be the mass per unit volume, then the mass of this elementary disc $=\pi y^2 \delta x \rho$.

\therefore If (\bar{x}, \bar{y}) be the required C.G. then $\bar{y}=0$ since the C.G. of the solid lies on x-axis, which is the axis of symmetry.



(Fig. 27)

$$\text{And } \bar{x} = \frac{\int x dm}{\int dm} = \frac{\int_{x=a}^b x \rho \pi y^2 dx}{\int_{x=a}^b \rho \pi y^2 dx} = \frac{\int_a^b xy^2 dx}{\int_a^b y^2 dx}$$

where $y=f(x)$ and $\rho=\text{constant}$.

(b) If the solid be generated by the revolution of the curve $x=f(y)$ lying between abscissae $y=a$ and $y=b$ about y-axis, then the mass of the elementary disc formed as above is $\pi x^2 \delta y \rho$, where ρ is the mass per unit volume. Also y-axis being the axis of symmetry, in this case we have $\bar{x}=0$, where (\bar{x}, \bar{y}) is the C.G. of the solid of revolution.

$$\text{And } \bar{y} = \frac{\int y dm}{\int dm} = \frac{\int_{y=a}^b y \rho \pi x^2 dy}{\int_{y=a}^b \rho \pi x^2 dy} = \frac{\int_a^b y x^2 dy}{\int_a^b x^2 dy}$$

where $x=f(y)$ and $\rho=\text{constant}$.

Solved Examples on § 8

*Ex. 1. Find the C.G. of a solid right circular cone.

(Avadh 88 ; Garhwal 89 ; Gorakhpur 89, Kanpur 91,

Ranchi 85 ; Rohilkhand 91, 87)

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Solution. The solid right circular cone is generated by revolving the right angled triangle OAB about OA . Let O be the origin

and OA the x -axis. Then if $\angle BOA = \alpha$, the equation of OB is
 $y = x \tan \alpha$... (1)

Take two neighbouring points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ on the line OB . Then the mass of the elementary disc generated by the revolution of the area $PNMQ$ about x -axis = $\pi y^2 \delta x \rho$, where ρ is the mass per unit volume. Let $OA = h$ be the height of the cone.

The C. G. of this cone lies on OA , the axis of revolution. (Fig. 28)

A. If (\bar{x}, \bar{y}) be the required C.G. then $\bar{y} = 0$ and

$$\begin{aligned}\bar{x} &= \frac{\int x dm}{\int dm} = \frac{\int_{x=0}^h x \cdot \rho \pi y^2 dx}{\int_{x=0}^h \rho \pi y^2 dx} = \frac{\int_{x=0}^h x x^2 \tan^2 \alpha \cdot dx}{\int_{x=0}^h x^2 \tan^2 \alpha \cdot dx}, \text{ from (1)} \\ &= \frac{\int_0^h x^3 dx}{\int_0^h x^2 dx} = \frac{h^4/4}{h^3/3} = \frac{3h}{4}\end{aligned}$$

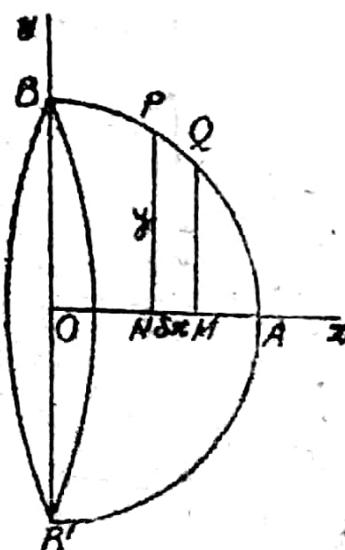
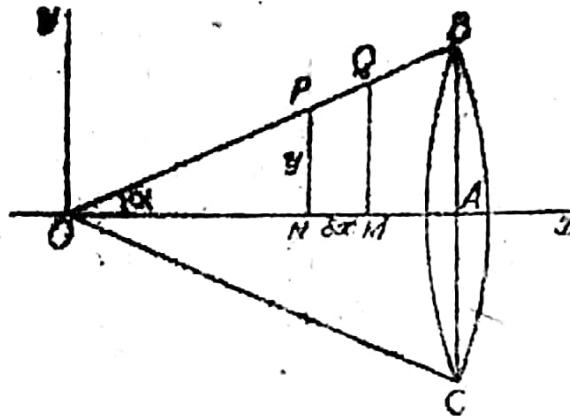
A. C.G. of a solid right circular cone of height h lies on its axis at a distance $\frac{3}{4}h$ from the vertex. Ans.

~~MII 71~~ Ex. 2. Find the C. G. of a solid uniform hemisphere of radius a . (Delhi 85 ; Garhwal 85)

Solution. The quadrant AB of a circle with O as centre and radius a , when revolved about one of the bounding radii OA generates the hemisphere. Take O as origin and OA as x -axis, then OB is y -axis.

\therefore The equation of the arc AB of the circle is $x^2 + y^2 = a^2$.

Take two neighbouring points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ on the circle. From P and Q draw PN and QM perpendiculars to x -axis. If this area $PNMQ$ is revolved about x -axis it would generate the elementary disc whose mass = $\delta m = \rho \pi y^2 \delta x$, where ρ is mass per unit volume. Also the C. G. of this elementary disc may be taken as N



(Fig. 29)

$(x, 0)$ as $NM = \delta x$ is very small. Also for the hemisphere x -axis is the axis of symmetry $\therefore \bar{y} = 0$, where (\bar{x}, \bar{y}) is the required C.G. of the hemisphere.

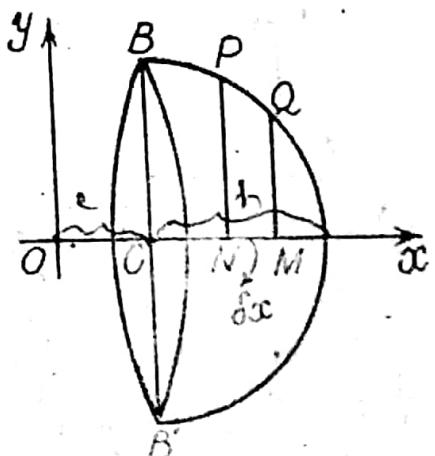
$$\begin{aligned} \text{And } \bar{x} &= \frac{\int x dm}{\int dm} = \frac{\int_{x=0}^a x \rho \pi y^2 dx}{\int_{x=0}^a \rho \pi y^2 dx} = \frac{\int_0^a xy^2 dx}{\int_0^a y^2 dx} \\ &= \frac{\int_0^a x(a^2 - x^2) dx}{\int_0^a (a^2 - x^2) dx}, \quad \forall x^2 + y^2 = a^2 \text{ or } y^2 = a^2 - x^2 \\ &= \frac{\left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a}{\left[a^2 x - \frac{x^3}{3} \right]_0^a} = \frac{\left(\frac{a^4}{2} - \frac{a^4}{4} \right)}{\left(a^3 - \frac{a^3}{3} \right)} = \frac{a^4}{4} \times \frac{3}{2a^3} = \frac{3a}{8} \end{aligned}$$

A C.G. of a hemisphere of radius a is a point on its axis at a distance $3a/8$ from the centre of the base of the hemisphere.

H2OS M110 M11 *Ex. 3. Find the C. G. of the segment of a sphere of radius a cut off by a plane at a distance c from the centre. (Gorakhpur 87)

Solution. Take the centre of the sphere as origin and the radius perpendicular to the given plane as the axis of x . Then as in last example we have $\bar{y} = 0$ and

$$\begin{aligned} \bar{x} &= \frac{\int x dm}{\int dm} = \frac{\int_{x=0}^a x \rho \pi y^2 dx}{\int_{x=0}^a \rho \pi y^2 dx} \\ \text{OR} \quad &= \frac{\int_0^a x(a^2 - x^2) dx}{\int_0^a (a^2 - x^2) dx}, \\ 2a^3 - 3a^2c + c^3 &= \int_0^a x(a^2 - x^2) dx \\ = 2a^3 - 2a^2c &= \frac{-a^2c + c^3}{\int_0^a (a^2 - x^2) dx} \\ - a^2c + c^3 &= \frac{\int_0^a (a^2 - x^2) dx}{a^2 - x^2} \\ - a^2c + c^3 &= \frac{2a^3(a - c) - c(a^2 - c^2)}{a^2 - c^2} \\ = (a - c)(2a^2 - c^2) &\because x^2 + y^2 = a^2 \text{ or } y^2 = a^2 - x^2 \\ = (a - c)(2a^2 - c^2) &= (a - c)(2a^2 - c^2) \\ - (a - c)(2a^2 - c^2) &= \frac{\left[\frac{a^3 x}{2} - \frac{x^4}{4} \right]_0^a}{\left[a^2 x - \frac{x^3}{3} \right]_0^a} = \frac{\left(\frac{a^4}{2} - \frac{a^4}{4} \right) - \left(\frac{a^3 c^2}{2} - \frac{c^4}{4} \right)}{\left(a^3 - \frac{a^3}{3} \right) - \left(a^3 c - \frac{c^3}{3} \right)} \\ = (a - c)(2a^2 - c^2) &= \frac{3(a - c)^2(a + c)^2}{4(2a + c)} \\ - (a - c)(2a^2 - c^2) &= \frac{3(a + c)^2}{4(2a + c)} \end{aligned}$$



(Fig. 30)

N.B. Put $c=a-h$, to obtain C.G. of a segment of height h .

C. G. of Solid of Revolution

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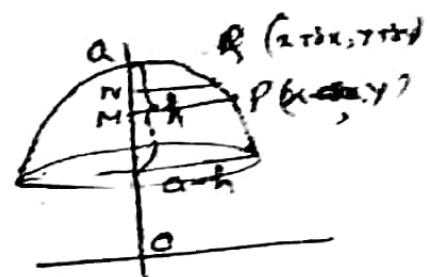
Hence the required C.G. is given by $\bar{x} = \frac{3(a+c)^2}{4(2a+c)}$, $\bar{y} = 0$. Ans.

*Note. Putting $c=0$, we can get the C.G. of a hemisphere.
Ex 4 (a) Find the C.G. of the volume formed by the revolution

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Find the CG of a segment of height h of a sphere of radius a, and deduce the position of the CG of a hemispherical shell.

Sol: Let us take the centre of the sphere as the origin and the radius perpendicular to the plane face w.r.t. the y-axis. Then the CG will be on y-axis. \therefore If (x, y) is the CG. Then



$x=0$. Let sections $P(x, y)$ & $Q(x+h, y+h)$ be two neighbouring N. on the circle due to the revolution of which about y-axis, the segment is formed. If area revolves about the y-axis, then a circular disc of vol. $\pi x^2 dy$ will be formed. If p be the mass per unit vol., then $\delta m = \pi x^2 y \cdot p$. The height of the segment is h . $\therefore y$ varies from $a-h$ to a .

$$\therefore \bar{y} = \frac{\int_a^a y dm}{\int dm} = \frac{\int_{a-h}^a y \pi x^2 p dy}{\int_{a-h}^a \pi x^2 p dy}$$

$$\begin{aligned}
 \bar{y} &= \frac{\int_{a-h}^a y \cdot (a-y) dy}{\int_{a-h}^a (a-y) dy} = \frac{\left[a^2 y - \frac{y^2}{2} \right]_{a-h}^a}{\left[a^2 y - \frac{y^3}{3} \right]_{a-h}^a} \\
 &= \frac{\frac{a^3}{2} - \frac{a^2}{4} - \frac{a^2}{2}(a-h)^2 + \frac{(a-h)^3}{3}}{a^2 \cdot a - \frac{a^3}{3} - a^2(a-h) + \frac{(a-h)^3}{3}} \\
 &= \frac{\frac{1}{4} \left(\frac{a^3}{2} + (a-h)^2 - 2a^2(a-h)^2 \right)}{\frac{1}{3} \left\{ 2a^3 - 3a^2(a-h) + (a-h)^3 \right\}} \\
 &= \frac{3}{4} \frac{\{a^2 - (a-h)^2\}^2}{2a^3 + a^2(a-h) - 4a^2(a-h) + (a-h)^3} \\
 &\Rightarrow \frac{3}{4} \frac{(a-a+h)^2 (a+a-h)^2}{a^2 (2a+a-h) - (a-h) \{ 4a^2 - (a-h)^2 \}} \\
 &= \frac{3}{4} \frac{h^2 (2a-h)^2}{(3a-h) \{ a^2 - 2a(a-h) + (a-h)^2 \}} \\
 &= \frac{3}{4} \frac{h^2 (2a-h)^2}{(3a-h) (a-a+h)} \\
 &= \frac{3}{4} \frac{h^2 (2a-h)^2}{(3a-h) h^2} \\
 &\Rightarrow \frac{3}{4} \frac{(2a-h)^2}{(3a-h)} \\
 \therefore \text{The C.G. is at } &\left(0, \frac{3}{4} \frac{(2a-h)^2}{(3a-h)} \right) \\
 \text{Deduction: For hemisphere, } h=a. & \\
 \therefore \text{C.G. is at } &\left(0, \frac{13}{4} \cdot \frac{(2a-a)^2}{(3a-a)} \right) \\
 \text{or at } &\left(0, \frac{3a}{8a} \right)
 \end{aligned}$$